Construction of Shortened Systematic PAC Codes Based on Monte-Carlo Algorithm

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***Abstract*—Polarization transform limits the code length of polarization-adjusted convolutional (PAC) codes to a power of 2, which hinders the flexible use of PAC codes. Shortening is an important method to change the length of the PAC codes. Systematic code makes it possible to select the shortened bit flexibly. In this paper, we propose a flexible code length scheme for shortened systematic PAC codes, where the proposed scheme generates frozen set and shortened set based on Monte-Carlo algorithm. Simulation results show that the Monte-Carlo algorithm can significantly improve the performance in frame error rate (FER), and the proposed shortening scheme can slightly improve the performance in FER.**

***Index Terms*—Polarization-adjusted convolutional codes, shortening, systematic encoding.**

I. Introduction

Polar codes are the capacity-achieving codes for symmetric binary-input memoryless channels proposed by Arikan in [1], where the successive cancellation (SC) decoder with low-complexity is used. However, the SC decoding algorithm for the polar codes of finite-length shows weak performance. On the one hand, the SC decoder is suboptimal. On the other hand, polar codes have a poor minimum distance. In [2], Arikan proposes cascading convolutional codes and polarization codes, and uses sequential decoders, i.e., polarization-adjusted convolutional (PAC) codes. PAC codes can better achieve capacity allocation and reduce capacity loss under finite-length codes. Due to the variable complexity of the Fano decoder, the SC list (SCL) [3] decoder for PAC codes would be more practical.

The code length of PAC/polar codes is only allowed to be a power of 2 due to the polarization matrix. The inflexibility of code length is a major drawback in the practical use of PAC/polar codes, which must be improved. Polar codes use puncturing and shortening to achieve arbitrary code length, which has the advantage of making decoding almost as complex as the mother code. The same decoder is used for puncturing and shortening, the difference being whether the decoder knows the non-transmit bit and sets the corresponding log-likelihood ratio (LLR) to zero and infinity. In general, shortening has good performance at high code rates, while puncturing has good performance at low code rates. In our work, we only consider shortening at high code rates. Systematic PAC/polar codes are consistent with the frame error rate (FER) performance of non-systematic codes, and the bit error rate (BER) performance is improved. Since the systematic codes are characterized by the fact that the information bit appears as a part of the codeword, the shortened bit can be encoded as a part of the information bit so that the decoder knows the shortened bits [4]. Reversal quasi-uniform puncturing (RQUP) in [5] and [6], a shortening rather than a puncturing scheme, has been used in fifth-generation mobile communications [7]. RQUP method for non-systematic codes is applied in systematic codes, which can lead to potential performance degradation. Specifically, the RQUP method of the polar codes is for the polarization transform matrix, and the systematic codes shortening method is for the systematic codes generation matrix.

The FER/BER of the PAC codes depends largely on the construction. In the SC decoder, the bit channels are sorted according to their reliability, and the most reliable channel is selected from them to transmit information bits. The Bhattacharyya parameter is used in [1] to measure the reliability of the bit channels. In [8], a Gaussian approximation (GA) is proposed to estimate the LLR of each bit. However, in SCL decoding, reliability ordering may not exist, i.e., there is no ordering that allows the construction of arbitrary code rates [9]. Moreover, among many scenarios, the impact of shortening on construction is not specifically considered. Recently, considering the effect of cutoff rate on sequential decoding complexity, a Monte-Carlo (MC) based construction of PAC codes was proposed. This method can limit the average complexity of sequential decoding [10]. Since Monte-Carlo construction focuses only on first bit error (FBE), it is a widely practical approach. It can adapt to various rates and code lengths, and is also suitable for various concatenated coding schemes. Shortened set and frozen set can be determined in a unified framework which is jointly designed by the shortening pattern and the frozen bit positions [11].

In this paper, our main contributions are as follows.

- The matrix form of the systematic PAC codes is given for further analysis of shortened bit selection.
- The Monte-Carlo construction is applied to the systematic PAC codes, and an improvement scheme based on the RQUP shortening scheme is proposed.

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The paper is organized as follows. Section II introduces the coding and shortening of the systematic PAC codes. Section III introduces the proposed optimization algorithm. Section IV shows the numerical results. Section V draws the conclusions.

All algebraic operations are conducted over spaces {0,1}. $\mathbf{v}_A$ denotes a subvector $(v_i, i \in \mathbf{A})$ and $\mathbf{G}_{\mathbf{A},\mathbf{B}}$ denotes rows $\mathbf{A}$ and columns $\mathbf{B}$ of the matrix $\mathbf{G}$. The cardinality of a set is denoted by $|\cdot|$.

II. Preliminaries

A. PAC Codes

The PAC codes have to be undergone convolutional operations and be undergone polarization transforms. The length of PAC codes can be denoted as $N_m = 2^n$. Let $\mathbf{v}, \mathbf{u} \in \{0,1\}^{N_m}$ represent the input of convolution operation, and the input of polarization transform, respectively. $\mathbf{A}$ represents the positions of K_m information bits and $\mathbf{A}^c$ represents the positions of $N_m - K_m$ frozen bits, and in general, the value of frozen bit is fixed to 0. Information bits $\mathbf{d} = (d_0, d_1, ...d_{K_m-1})$ are assigned into vector $\mathbf{v} = (v_0, v_1, ...v_{N_m-1})$, which is called rate-profiling. The convolution operation is expressed as

$$\mathbf{u} = \mathbf{v}\mathbf{T}, \tag{1}$$

where $\mathbf{T}$ is an upper-triangular Toeplitz matrix. It is obtained by cyclic shift of $g = [g_0, g_1, ..., g_c]$ with $g_i \in \{0,1\}, 1 \leq i \leq c$ and $g_0 = 1$. Sequential method implementations can be computed by

$$u_i = \sum_{j=0}^{c} g_j v_{i-j} = v_i + s_i, \tag{2}$$

where $s_i = g_1 v_{i-1} + g_2 v_{i-2} + ... + g_c v_{i-c}$. Then, the polarization transform can be calculated by

$$\mathbf{x} = \mathbf{u}\mathbf{L}, \tag{3}$$

where $\mathbf{L}$ is the generator matrix of polar codes. It is obtained by $\mathbf{F}^{\bigotimes n}$defined as the n-th Kronecker power of $\mathbf{F} \triangleq \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

B. Systematic PAC Codes

For the systematic encoding of PAC codes, the data words $\mathbf{d}$ appears as a part of the codewords $\mathbf{x}$, i.e, $\mathbf{x}_\mathbf{A} = \mathbf{d}$. Systematic codes can be generated based on vector and matrix multiplication, but complex arithmetic operations make the coding impossible to use in practice. *Proposition* 2 in [4] gives an encoding method for systematic codes with low-complexity. Under certain restrictions, the original encoder of PAC codes can be used to encode systematic codes. It can be calculated by three steps

- $\mathbf{u}_\mathbf{A} = \mathbf{d}\mathbf{L}_{\mathbf{A},\mathbf{A}}, \mathbf{v}_{\mathbf{A},\mathbf{A}^c} = \mathbf{f}$.
- Calculation of $\mathbf{u}_{\mathbf{A}^c}$ and $\mathbf{v}_\mathbf{A}$ by sequential method
$$\begin{cases} v_i = u_i + s_i, & i \in \mathbf{A} \\ u_i = v_i + s_i, & i \in \mathbf{A}^c \end{cases}$$
and updates s_i.
- Calculate $\mathbf{x}$ using (3).

In order to use polar codes encoders, the above process must satisfy $\mathbf{L}_{\mathbf{A}^c,\mathbf{A}} = \mathbf{0}$ and $L_{\mathbf{A},\mathbf{A}} = L^{-1}_{\mathbf{A},\mathbf{A}}$.

The generation of codeword $\mathbf{x}$ is represented as

$$\mathbf{x} = \mathbf{d}\mathbf{G}. \tag{4}$$

Definition of matrix $\mathbf{E}(\mathbf{A})$ [12] is

$$\mathbf{E}(\mathbf{A}) = (E_{i,j})_{i=0}^{|\mathbf{A}|-1}{}_{j=0}^{N_m-1}, \tag{5}$$

where $E_{i,j} = \begin{cases} 1, & \text{if} \quad j = A_i \\ 0, & otherwise \end{cases}$ $and \quad A_0 < A_1 < ... < A_{|\mathbf{A}|-1}$.

Proposition 1: Given information set $\mathbf{A}$. If $\mathbf{f} = \mathbf{0}$, the generation matrix of systematic PAC codes is (For brevity, $\mathbf{A}$ in $\mathbf{E}(\mathbf{A})$ has been omitted)

$$\mathbf{G}(\mathbf{A}) = \mathbf{E}\mathbf{L}\mathbf{E}^T(\mathbf{E}\mathbf{T}\mathbf{E}^T)^{-1}\mathbf{E}\mathbf{T}\mathbf{L}. \tag{6}$$

Proof: Observe (1) and (3), we need to prove that $\mathbf{d}\mathbf{E}\mathbf{L}\mathbf{E}^T(\mathbf{E}\mathbf{T}\mathbf{E}^T)^{-1}\mathbf{E} = \mathbf{v}$. We have

$$\mathbf{u}_\mathbf{A} = \mathbf{v}_\mathbf{A}\mathbf{T}_{\mathbf{A},\mathbf{A}} + \mathbf{v}_{\mathbf{A}^c}\mathbf{T}_{\mathbf{A}^c,\mathbf{A}} = \mathbf{v}_\mathbf{A}\mathbf{T}_{\mathbf{A},\mathbf{A}}$$

since $\mathbf{v}_{\mathbf{A}^c} = \mathbf{0}$. Thus $\mathbf{v}_\mathbf{A} = \mathbf{u}_\mathbf{A}\mathbf{T}^{-1}_{\mathbf{A},\mathbf{A}}$ ($g_0 = 1$ and $\mathbf{T}$ is the upper-triangular matrix, hence, $\mathbf{T}_{\mathbf{A},\mathbf{A}}$ is invertible.) Due to the property of matrix $\mathbf{E}$, $\mathbf{L}_{\mathbf{A},\mathbf{A}} = \mathbf{E}\mathbf{L}\mathbf{E}^T, \mathbf{T}_{\mathbf{A},\mathbf{A}} = \mathbf{E}\mathbf{T}\mathbf{E}^T$. Since $\mathbf{v}_{\mathbf{A}^c} = \mathbf{0}$, $\mathbf{v} = \mathbf{v}_\mathbf{A}\mathbf{E}$. Thus

$$\begin{aligned} \mathbf{v} &= \mathbf{v}_A\mathbf{E} = \mathbf{u}_\mathbf{A}\mathbf{T}^{-1}_{\mathbf{A},\mathbf{A}}\mathbf{E} \\ &= \mathbf{d}\mathbf{L}_{\mathbf{A},\mathbf{A}}\mathbf{T}^{-1}_{\mathbf{A},\mathbf{A}}\mathbf{E} \\ &= \mathbf{d}\mathbf{E}\mathbf{L}\mathbf{E}^T(\mathbf{E}\mathbf{T}\mathbf{E}^T)^{-1}\mathbf{E} \end{aligned}$$

C. Shortening of Systematic PAC Codes

For shortening, $|\mathbf{S}|$ bits of the codeword $\mathbf{x}$ are limited to the fixed values, such as zero, i.e., $\mathbf{x}_\mathbf{S} = \mathbf{0}$, where $\mathbf{S}$ is called the shortening set, and $K = K_m - |\mathbf{S}|$. The code rate is R=K/N, where $N = N_m - |\mathbf{S}|$. For Systematic codes, $\mathbf{x}_\mathbf{A} = (\mathbf{d}, \mathbf{0}) = \mathbf{d}'$, where $\mathbf{A} = \mathbf{A} \cup \mathbf{S}$ and $|\mathbf{A}| = |\mathbf{d}| = K$. We have $\mathbf{x} = \mathbf{d}'\mathbf{G}(\mathbf{A}')$.

Define the set $\mathcal{I}(b)$ as

$$\mathcal{I}(b) = \{j \in [0 : N_m - 1] : |\mathcal{Q}(g_{:,j}) \setminus \mathbf{S}| = b\}, \tag{7}$$

where $b \in \{0,1\}$, $g_{:,j}$ denotes column j of the matrix and $Q(g_{:,j})$ [5] denotes the index set of 1 position in $g_{:,j}$. In order to avoid additional complexity, we only consider $\mathcal{I}(0)$, $\mathcal{I}(1) \subset \mathbf{A}'$ and $|\mathcal{I}(0)| = |\mathbf{S}|$, $|\mathcal{I}(1)| = K$.

For example, $N_m = 8$, $K_m = 6$, $g = [1,0,1,1,0,1,1]$, and $\mathbf{A}' = \{2,3,4,5,6,7\}$. By *Proposition* 1, we have

$$\mathbf{G}(\mathbf{A}^{'})=\begin{bmatrix}1&0&1&0&0&0&0&0\\0&1&0&1&0&0&0&0\\1&0&0&0&1&0&0&0\\0&1&0&0&0&1&0&0\\1&0&0&0&0&0&1&0\\0&1&0&0&0&0&0&1\end{bmatrix}. \tag{8}$$

If $N=5$, $\mathbf{S}=\{2,4,6\}$ is not considered. Since $\mathcal{I}(0)=\{0,2,4,6\}\not\subset\mathbf{A}^{'}$ and $|\mathcal{I}(0)|=4\neq 3$. If $N=6$, $\mathbf{S}=\{2,4\}$ is not considered. Since $\mathcal{I}(1)=\{0,3,5,6,7\}\not\subset\mathbf{A}^{'}$ and $|\mathcal{I}(1)|=5\neq 4$.

III. Proposed Shortening Scheme

This section briefly introduces the Monte-Carlo construction and the RQUP scheme, and then proposes a new shortening scheme of systematic codes for the Monte-Carlo construction.

A. Monte-Carlo Construction

Construction can be regarded as the process of selecting (N_m-K_m) error prone bits at N_m locations and fixing their values. The event of the occurrence of sequential decoding bit error at the i-th bit is $\mathcal{B}_i=\left\{(\mathbf{u},\mathbf{y}):\hat{\mathbf{u}}_{[0:i-1]}=\mathbf{u}_{[0:i-1]},\hat{u}_i\neq u_i\right\}$, where $\mathbf{y}$ is the output of the channel. The block error event $\mathcal{E}$ is the union of $\mathcal{B}_i$ events. Therefore, the probability of $\mathcal{E}$ can be expressed as

$$\mathrm{P}(\mathcal{E})=\sum_{i\in\mathcal{A}}\mathrm{P}\left(\mathcal{B}_i\right). \tag{9}$$

The FBE is critical in construction [13]. Monte-Carlo scheme only focuses on FBE. Its index is $j=\min(i):v_i\neq\hat{v}_i$. Under the selected signa-to-noise ratio (SNR), multiple decoding is performed to count the first error number of each bit. Then the bits with the most errors for the first time can be removed from the precomputed information set. Repeat this process until N_m-K_m frozen bits are selected.

B. RQUP Scheme

When using RQUP, the shortened set first selects the last $|\mathbf{S}|$ bit, which is $[N_m-|\mathbf{S}|:N_m-1]$, and then performs bit-reversal permutation. Bit permutation causes the shortened positions to be roughly uniform in the codeword, i.e., the distance between any two neighboring shortened positions is approximately the same [6]. Note that if the bit permutation matrix $\mathbf{B}_{N_m}$ is considered, (3) can be modified as

$$\mathbf{x}=\mathbf{u}\mathbf{B}_{N_m}\mathbf{L}. \tag{10}$$

$Proposition$ 2: RQUP algorithm for systematic PAC codes ensures that the decoder knows every bit in $u_i, i\in[N_m-|\mathbf{S}|:N_m-1]$.

$Proof$: Since $\mathbf{L}^{-1}=\mathbf{L}$, equation (3) can be written as $\mathbf{u}=\mathbf{x}\mathbf{L}$. Rewrite $Theorem10$ in [6]: Since the lower-triangle property of the matrix $\mathbf{L}$, $u_i=\sum_{i\leqslant j\leqslant N_m-1,\ \mathbf{L}_{i,j}=1}x_j,\ N_m-|\mathbf{S}|\leqslant i\leqslant N_m-1$. If each bit $x_j,\ j\in[N_m-|\mathbf{S}|:N_m-1]$ is set to a fixed value, each bit u_i will be set to a fixed value.

C. Improved Algorithm

The construction of frozen set and shortened set can be considered as an optimization problem. The complete set $\{0,1,...,N_m-1\}$ is divided into three sets, namely, information set, frozen set, and shortened set, to minimize BER, FER, etc. Without losing generality, we pay more attention to FER. This optimization problem is given by

$$\begin{aligned}(\mathbf{A}_{opt},\mathbf{S}_{opt})=&\underset{\mathbf{A},\mathbf{S}\subset\{0,1,\dots,N_m-1\}}{\arg\ \min}\ FER\\ s.t.\quad&|\mathbf{S}|=N_m-K_m\\&|\mathbf{A}|=K_m\\&\mathbf{A}\cup\mathbf{S}=\emptyset.\end{aligned} \tag{11}$$

To satisfy the low-complexity implementation of systematic PAC codes, the selection of the frozen bit is only according to the conditions mentioned in the Section II. The number of first errors for each bit is obtained by Monte-Carlo simulation and sorted. As shown in $algorithm$ 1 function MC, where $\mathbf{A}_{in}$ and $\mathbf{S}_{in}$ are the initial values of the information set and the shortened set, respectively. In the absence of their initial values, they are complete set and empty set, respectively. *M* and *P* are the vectors composed of 0 and 1, where the 1s imply the information or shortened positions. Function $MonteCarlo$ returns the number of FBE and FER at various code rates. For more details of function $MonteCarlo$, we refer to [10]. Considering shortening, we skip the process of predicting the set of information. The function $argsort$ returns the index sorted in ascending order. $\mathcal{Q}^{-1}(\cdot)$ is the inverse transformation of $\mathcal{Q}(\cdot)$.

$Proposition$ 3: If the systematic PAC codes are shortened using RQUP algorithm, $v_i, i\in[N_m-|\mathbf{S}|:N_m-1]$ is not FBE.

$Proof$: Assume $i\in[N_m-|\mathbf{S}|:N_m-1]$ and v_i is FBE. From (2), we have $v_i=u_i+s_i\neq\hat{v}_i=\hat{u}_i+\hat{s}_i$. From $Proposition$ 2, u_i and $\hat{u}_i$ are know, i.e., $u_i=\hat{u}_i$, then $s_i\neq\hat{s}_i$. We have $s_i+\hat{s}_i=\sum_{k\in D}g_k(v_{i-k}+\hat{v}_{i-k})=1$, where $D=\{k\in[1,c]:g_k\neq 0\}$. This indicates that there is at least one pair of $(v_{i-k},\ \hat{v}_{i-k})$, $v_{i-k}\neq\hat{v}_{i-k}$ $(i-k<i)$, which is inconsistent with the assumption that v_i is FBE.

$Proposition$ 3 shows that the RQUP algorithm fixes the shortened set in optimization problem (11), because of the Monte-Carlo construction only selects FBE. In other words, the RQUP shortening set protects some bits from being selected in the frozen set, which has potential performance implications. An alternative algorithm for RQUP is presented in function $shorten$ of $algorithm$ 1, where $mllrs$ is the reliability of each bit. In order to maintain the characteristics of quasi-uniform shortening, each shortened bit was selected at each interval. Due to the limitation of the coding method of the systematic codes, the shortened bits cannot belong to the frozen set. Therefore, one selection does not ensure that enough bits can be selected, and multiple selections must be made. Note that the characteristic of RQUP is in the case of bit permutation,

Algorithm 1: Construction of Shortened Systematic PAC Codes

Input: SNR, N, K, g
Output: **A**, **S**
$N_m \leftarrow 2^{\lceil \log_2 N \rceil} \quad K_m \leftarrow K + N_m - N$
$i_b \leftarrow N_m - K_m + 1$
$(M, P, FER) \leftarrow MC(\{0, 1...N_m - 1\}, \emptyset)$
for $b \leftarrow N_m - K_m \quad to \quad 1$ **then**
  if $FER[b-1] < FER[b] < FER[b+1]$ **do**
    $i_b \leftarrow b$
    $break$
$(M', P', FER') \leftarrow MC(\mathcal{Q}(M[i_b]), \mathcal{Q}(P[i_b]))$
$in1 \leftarrow argmin(FER)$
$in2 \leftarrow argmin(FER')$
if $FER[in1] < FER'[in2]$ **then**
  $\mathbf{A} \leftarrow \mathcal{Q}(M) \quad \mathbf{S} \leftarrow \mathcal{Q}(P)$
else
  $\mathbf{A} \leftarrow \mathcal{Q}(M') \quad \mathbf{S} \leftarrow \mathcal{Q}(P')$
Function MC $(\mathbf{A}_{in}, \mathbf{S}_{in})$
  $i \leftarrow 0 \quad \mathbf{S} \leftarrow \mathbf{S}_{in}$
  $FER[0 : |A_{in}| - K_m] \leftarrow \{0\}$
  $M[0 : |A_{in}| - K_m][0 : N_m - 1] \leftarrow \{0\}$
  $M[i][0 : N_m - 1] \leftarrow \mathcal{Q}^{-1}(\mathbf{A}_{in})$
  $P[0 : |A_{in}| - K_m][0 : N_m - 1] \leftarrow \{0\}$
  $P[i][0 : N_m - 1] \leftarrow \mathcal{Q}^{-1}(\mathbf{S}_{in})$
  while $|\mathcal{Q}(M[i])| + N_m - N > K_m$ **do**
    if $\mathbf{S}_{in} \neq \emptyset$ **then**
      $\mathbf{S} \leftarrow shorten(\mathcal{Q}(M[i]) \cup \mathcal{Q}(P[i]))$
    $P[i+1][0 : N_m - 1] \leftarrow \mathcal{Q}^{-1}(\mathbf{S})$
    $(h, FER[i]) \leftarrow MonteCarlo(\mathcal{Q}(M[i]), \mathbf{S})$
    $J \leftarrow argsort(-h)$
    for $j \in J$ **do**
      $\mathbf{B} \leftarrow \mathcal{Q}(M[i]) \cup \mathbf{S} \setminus j$
      if $L_{\mathbf{B},\mathbf{B}} L_{\mathbf{B},\mathbf{B}} = I_{|\mathbf{B}|}$ and $L_{\mathbf{B}^c,\mathbf{B}} = 0$ **then**
        $M[i+1] \leftarrow \mathcal{Q}^{-1}(\mathbf{B} \setminus \mathbf{S})$
        $i \leftarrow i + 1$
        $break$
  $(h, FER[i]) \leftarrow MonteCarlo(\mathcal{Q}(M[i]), \mathbf{S}, g)$
  return (M, P, FER)
Function $shorten$ (**C**)
  $\overleftarrow{\mathbf{S}} \leftarrow \emptyset$
  while$(|\mathbf{S}| < N_m - N)$ **do**
    $step \leftarrow \lceil \log 2(N_m / (N_m - N - |\overleftarrow{\mathbf{S}}|)) \rceil$
    for $i \leftarrow N_m / step$ to 1 **do**
      $i_0 \leftarrow (i-1) * step$
      $i_1 \leftarrow i * step - 1$
      $\overleftarrow{llrs} \leftarrow argsort(m\overleftarrow{llrs}[i_0 : i_1])$
      for $j \in \overleftarrow{llrs}$ **do**
        if $i_0 + j \in \overleftarrow{\mathbf{C}}$ **then**
          $\overleftarrow{\mathbf{S}} \cup \{i_0 + j\}$
          $break$
  return S

so our scheme also considers bit permutation when selecting bit. "$\leftarrow$" in the $algorithm$ 1 represents this process and the final returned result is inversely transformed. In the proposed shortening scheme, the frozen bit takes precedence over the shortened bit for selection. However, this inequality may result in performance degradation. For each frozen bit selection, we record its FER. The frozen set corresponding to the last extreme point of the FER is used as the initial value, and the Monte-Carlo construction is performed again. In our work, we use SC and SCL decoder for Monte-Carlo simulation, specifically

$$\begin{cases} \mathcal{L} \quad SC, & i \in H, \\ SCL, & i \in \{0, 1, ..., N_m - 1\} \setminus H, \end{cases} \tag{12}$$

where $H = \{i \in [0 : N_m - 1] : |\lambda_0^i| = \infty\}$ can be calculated by $algorithm\ A$ in [14] and λ_0^i is the i-th bit of 0 stage of the factor graph.

IV. Numerical Results

In this section, we give the FER performance of the systematic shortening codes. The reliability is calculated using ensity evolution with Gaussian approximation (DEGA), and the LLRs of all coded bits are set to the values corresponding to the design SNR of additive white Gaussian noise (AWGN) channel. For all shortened scenarios, SC and SCL decoders are used, and the list size L=32. The codes involved in transmission is modulated by binary phase shift keying (BPSK) and transmitted on AWGN channel. Convolution polynomial g is 0o133 in octal format and c=6. The maximum number of iterations is 10^7, and the maximum number of error frames is 10^2.

For comparison, the performance in [15] is also considered. The systematic PAC codes are implemented with low complexity, and $\mathbf{S} \subsetneqq \mathcal{P}_{[N-K:N_m-1]}$, where $\mathcal{P}$ is the indices sequence of N_m polarised subchannels in ascending order of reliability.

Fig. 1 shows the performance of PAC codes with N=96 and $|\mathbf{S}|$=32. We have the following observations. For code rates R=0.5 and 0.75, Monte-Carlo construction is better than DEGA construction. When Monte Carlo construction is used, the proposed scheme has a performance advantage at a code rate of R=0.5, and its performance is close to that of the RQUP scheme at a code rate of R=0.75. Fig. 2 shows the performance of PAC codes with N=192 and $|\mathbf{S}|$=64. The results show that when the code length is 192, the Monte-Carlo construction is better than DEGA. Monte-Carlo construction can improve the performance of systematic PAC codes. For R=0.75, the proposed scheme has performance advantages at high SNR and has performance disadvantages at low SNR. For R=0.5, the proposed scheme has performance close to RQUP.

Fig. 3 is the FER performance change during the Monte-Carlo construction process. It can be observed that in most cases, the FER performance improves as the code rate drops. However, the FER performance occasionally deteriorates with the drop of code rate. Therefore, it is necessary to fix the shortening set and then update the frozen set in the proposed shortening scheme.

V. Conclusion

In this study, we proposed the joint constructions of shortened set and frozen set of PAC codes with flexible codes length and code rates. Monte-Carlo construction method used for shortened systematic PAC codes. Then, we further proposed

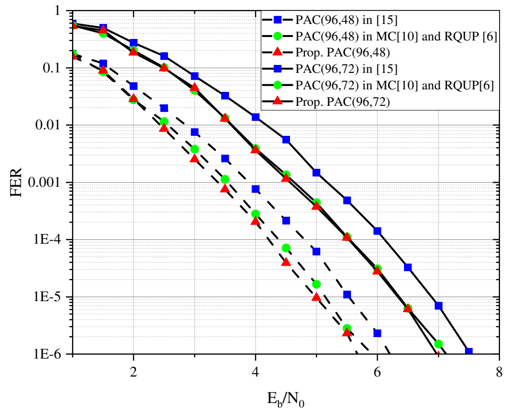

Fig. 1. FER performance with the PAC codes for N=96, $|\mathbf{S}| = 32$ in AWGN channels.

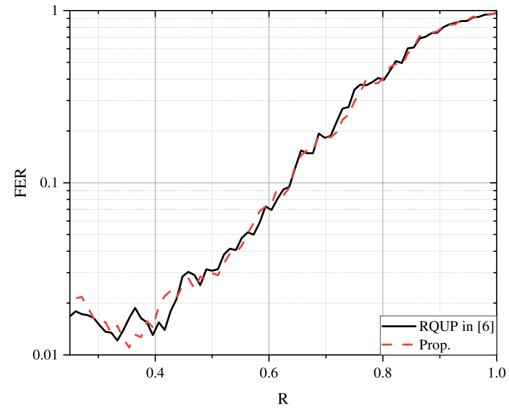

Fig. 3. FER performance at various code rates for N=96, $|\mathbf{S}| = 32$ during construction.

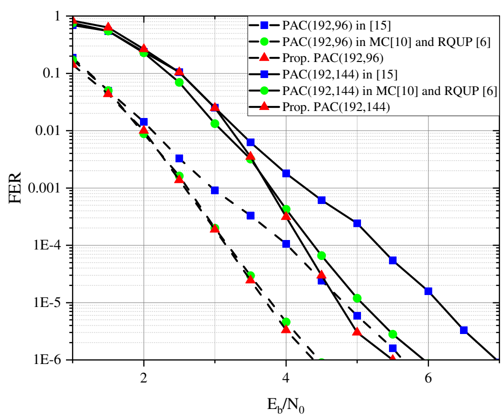

Fig. 2. FER performance with the PAC codes for N=192, $|\mathbf{S}| = 64$ in AWGN channels.

a scheme to replace the RQUP scheme in the systematic PAC codes and maintain the the characteristics of quasi-uniform. The proposed scheme dynamically selects the shortened set, and is comparable to RQUP scheme. The construction method considering only FBE is not applicable to RQUP scheme and there may be other uniform shortening schemes that improve performance. Under the RQUP algorithm, another construction method should be proposed for the systematic PAC codes.

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